Method Maximizing the Spread of Influence in Directed Signed Weighted Graphs

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Abstract. *We propose a new method for maximizing the spread of influence, based on the identification of significant factors of the total energy of a control system. The model of a socio-economic system can be represented in the form of cognitive maps that are directed signed weighted graphs with cause-and-effect relationships and cycles. Identification and selection of target factors and effective control factors of a system is carried out as a solution to the optimal control problem. The influences are determined by the solution to optimization problem of maximizing the objective function, leading to matrix symmetrization. The gear-ratio symmetrization is based on computing the similarity extent of fan-beam structures of the influence spread of vertices v_i and v_j to all other vertices. This approach provides the real computational domain and correctness of solving the optimal control problem. In addition, it does not impose requirements for graphs to be ordering relationships, to have a matrix of special type or to fulfill stability conditions. In this paper, determination of new metrics of vertices, indicating and estimating the extent and the ability to effectively control, are likewise offered. Additionally, we provide experimental results over real cognitive models in support.*

Keywords

Directed weighted graphs, gear-ratio symmetrization, maximizing the spread of influence

1. Introduction

In underdetermined poorly formalized environments i.e. management or social and economic systems it is difficult or sometimes impossible to identify correctly a specific problem in the control system because real-world systems contain a big amount of factors and relationships between them that influence a system functioning. Furthermore, different types of limitations (financial, technological, political, etc.) make this task even more complicated. In this case, a directionality of system work becomes very important. Misconceptions about system function due to its complexity and multiplicity of feedbacks lead to an erroneous decision-making process. Therefore, selection of the main factors that significantly influence a system is an urgent problem.

A complex model of such a poorly formalized system can be represented as a fuzzy oriented weighted cyclic signed graph with cause-effect relationships, which may be formed by several methods, including the use of fuzzy cognitive maps [1]. This graphical representation of the behavioral model of systems helps to provide a general overview of factors influence on each other. Nevertheless, the system target direction in addition to selection of key elements from the system remains difficult to achieve. This obstacle is observed due to the large dimensionality of graphs and the presence of numerous feedbacks (the complexity of concepts and relations between them). Moreover, dimensions of modern graphs exceed the capacity of human perception, so their standard representation loses its informative value.

For instance, in case that vertices are assigned arbitrarily (by experts), a decision maker may assume that by increase of value of factors (estimated as bearing ones), the desired values of target factors will be reached as well. However, the system may act despite the decision maker's views. As a result, the main goal will not be achieved, and may even become estranged. Then a selection of essential factors may become the base for a verification of initial representation of a system. Comparison of the expected and real results of the work will allow to ensure the correctness of a system construction, and on this basis, to determine the place of application of control impacts. For instance, by forming a graph of corporate governance, the experts may assume that a significant component of the system is the transparency of the Board of Directors work. However, the automation results of the used system may reveal another key factor - the formation of a quality management system. Then, the control impacts have to be necessarily applied to other groups of factors, since the main objective is not to improve the work of the Board of Directors, but the corporate governance improvement, which is expressed in the respective graph. A comparison of the expectations and the automation results gives information about the reliability of the original graph of the system, or about necessity in the system representation changes. Accordingly, an approach for the selection of significant (key) factors of a system is a necessary and urgent management task.

Significant factors can be divided into two main groups: the impact factors (application control powers) and target factors (response) to the greatest extent varying under the power of their influence. To solve the problem of identifying and selecting the key factors presented in the form of vertices of the graph, the concept of effective controls and theory of effective controls are introduced in this article. Definitions of some metrics of the vertices states, reflecting the impact of effective control components in the control system, are described as well. This approach is based on the postulates of general system theory and on systems analysis applied to socio-economic systems control.

The novelty of the approach lies in the adaptation of control model in technical systems for the control in socio-economic systems. Accordingly, in the technical system control model, the growth vector of the target vertices parameters is determined by a known vector of control impacts from the solution to the system of linear algebraic equations (SLE). For a graph of socio-economic systems, the formulated task is to find a vector of control impacts, leading to a maximum growth of indices of target factors. At the same time, control factors and target factors are determined from the solution to SLE, rather than appointed by the decision-makers (the experts).

The decision is based on the model of impacts transition on the arcs of graph. Overcoming the possible instability during the transmission of impacts is achieved by appropriate formulation of the optimization problem. The problems of the spectrum location inside the unit circle and of the interpretation of complex values of the spectrum components are solved by their reduction to the optimization problem. This convergence of emerging iterative processes for the resolvent is guaranteed.

2. Related Work

Fuzzy cognitive maps are directed signed weighted cyclic graphs with cause-and-effect relationships. In general, the trends of graph analysis are connected with clustering, enlargement and metrics analysis of the graph.

In the studies conducted in [2], [3], [4] and [5], metrics, such as the node degree, betweenness centrality and proximity are developed for the analysis of social network, which are presented as unweighted graphs. The introduced measure of h-type (h-index, h-degree) [6] has potential in the field of analysis for weighted networks as compared with traditional indicators for unweighted ones. On the other hand, these approaches, based on the orientation of arcs, ignore their weight characteristics, as well as the influence transmitted from one vertex to another.

Zhang et al. [7] has attempted to take into account the influence effect of one vertex to another. Additionally, the authors proposed a new metric of the influence effect, which represents the product of the arcs' weights. The presented in the study method was designed for clustering graphs, adjacency matrices of which have a pronounced block structure. However, these computations are based on simple expressions having intuitive nature, which do not take into account the influence effect of the transitive transfer. In the articles [8] and [9], the degree of influence is not determined from intuitive considerations, but from solving the system of linear algebraic equations (SLE). These processes are considered in the dynamics, therefore, in the study [8] the conditions of stability become relevant, which was expressed in the restriction that the entire spectrum of the matrix must be located inside the unit circle while eigenvalues and eigenvectors appear to be complex.

Liu et al. [9] have analyzed the method of clustering CSIP based on the sorting of influence power, as the dynamic force of the impact between nodes, considering the surroundings. This method introduces new metrics of vertices, which can serve as the basis for the selection of key vertices. However, the results of the aforementioned study [9] were mainly focused on the stochastic matrix. This type of matrix is an adjacency matrix, in which the sums in the columns are strictly equal to one and the elements are non-negative. Failure to comply with these conditions makes the use of these methods practically impossible. In our study, we focus on the problems oriented to the general matrices rather than the special type, for the solution of which the implementation of sustainability conditions is not an essential principal requirement.

Furthermore, there is number of studies aimed at solving the problem of identifying the most influential nodes in the directed networks [10], [11], [12], [13] and [14]. In the studies [11], [12] and [13], the used approach involves unidirectional irreversible switching from one state to another. Thus, the active vertex may or may not cause the activation of dormant neighbors, but it cannot be returned to the inactive state. Similarly, in [12], the buyer making a purchase cannot go into a state of "a buyer who has not bought anything".

In the method proposed in [11] and [13], the existence of feedback is not permitted as opposed to a social network or cognitive model. That is, the vertex which has become active, cannot turn into more (or less) active state. The buyers of the product [12], which persuade their friends, will not become the buyers of the second instance of the product due to a positive feedbacks of their friends. In the socio-economic networks considered by us, this is not applicable. Agitator campaigns more actively, meeting an interested response from their audience, while he/she reduces activity in case of not finding an interest. Moreover, in the same network not only more or less active agitators might exist, but agitators having an opposite view, whose activity is also competitive with respect to the first point of view. In the method [11] and [13], the activation of inactive neighboring vertices may increase or decrease, but the process cannot take a negative value, in contrast to our case.

In addition, the methods [10], [11], [12] and [13] did not take into account the causal relationship between the vertices as an independent characteristic, but rather the similarity measure based on the topological characteristics of the network. Marinazzo et al. [14] applies Kernel-Granger causality method of multidimensional time series processing, including the networks containing cycles. Network models with a sufficiently large number of nodes (N) will contain more links (N^2), for optimal estimation of which time series, even of greater extent (kN^2), are required. In our case, time series are simply unavailable, making it impossible to apply this method [14]. Therefore, the respective characteristics are given by experts, and to minimize the error of the expert setting for the model parameters, the corresponding optimization problem is being solved as well. Thus, this approach cannot be used for tasks with cyclic signed directed graphs, as in our case.

3. Methods

Let's consider the solution to a problem from the perspective of these three approaches: traditional technical control system Subsec. 3.1. , control features in socio-economic systems, and presentation of the socio-economic system as a graph Subsec. 3.2.

3.1. The Mathematical Formulation of the Problem in Traditional Technical Control System

Consider a finite graph $G = \langle V, E\rangle$, where $V = \{v_1, v_2, ..., v_n\}$ is the finite set of vertices, n is the number of vertices, $E = \{\langle v_j, v_k\rangle | v_j \in V, v_k \in V\}$ is the finite set of arcs. The graph G corresponds to the adjacency matrix $\mathbf{A} = \|a_{jk}\|_{nxn}$, where weight a_{jk} of arc $\langle v_j, v_k\rangle$ can express the presence of the arc (Boolean value), the multiplicity of the arc, arc weight, fuzzy measure of the adjacency of vertices $v_j$and v_k, etc. Consider the following control model:

$$\begin{aligned} x_j &= u_j + \delta(a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{nj}x_n), \\ &= u_j + \delta\sum_{i=1}^{n} a_{ij}x_i, \end{aligned} \tag{1}$$

where x_j is the value of the growth rate associated with the vertex v_i; u_j is the amount of the controlling impact on the coefficient associated with the vertex v_j; δ is the damping factor, $0 < \delta \leq 1$. The model may be represented in the matrix form:

$$\vec{x} = \vec{u} + \delta\mathbf{A}^T\vec{x} \; or \; (\mathbf{E} - \delta\mathbf{A}^T)\vec{x} = \vec{u}, \tag{2}$$

where $\vec{x} = (x_1 x_2 \dots x_n)^T$ is the vector of growth rates associated with the vertices; $\vec{u} = (u_1 u_2 \dots u_n)^T$ is the vector of control impacts; $\mathbf{E}$ is the identity matrix.

The gain of the model can be explained as follows. With the direct effect of the control influence $\vec{u}$ on the $\vec{x}$:

$$\vec{x} = \vec{u}. \tag{3}$$

Taking into account the one-step spread of the impact on the graph arcs:

$$\vec{x} = \vec{u} + \mathbf{A}^T\vec{u}. \tag{4}$$

Taking into account the one-step spread of the impact on the graph arcs:

$$\vec{x} = \vec{u} + \mathbf{A}^T\vec{u} + \mathbf{A}^T\mathbf{A}^T\vec{u}. \tag{5}$$

On condition of the convergence of a forming series we have:

$$\vec{x} = \vec{u} + \mathbf{A}^T\vec{u} + \mathbf{A}^T\mathbf{A}^T\vec{u} + ... + \underbrace{\mathbf{A}^T\mathbf{A}^T...\mathbf{A}^T}_{k}\vec{u} + ... \quad (6)$$

To control the convergence we introduce the decrement δ:

$$\vec{x} = \vec{u} + \delta\mathbf{A}^T\vec{u} + \mathbf{A}^T\mathbf{A}^T\vec{u} + ... + \underbrace{\mathbf{A}^T\mathbf{A}^T...\mathbf{A}^T}_{k}u + ... \quad (7)$$

With absolute convergence we acquire the model Eq. (2). Note that the model Eq. (2) contains the final (not the dynamically changing) limit values of the growth parameters and impacts, although it can be interpreted in a first way as well. However, in the dynamic case it is better to use the following view:

$$\vec{x}^{(k+1)} = \vec{u}^{(k)} + \delta\mathbf{A}^T\vec{x}^{(k)}, \quad (8)$$

or

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + \vec{u}^{(k)} - (\mathbf{E} - \delta\mathbf{A}^T)\vec{x}^{(k)}, \quad (9)$$

which corresponds to the canonical form of iterative numerical schemes.

If the inverse value δ^{-1} belongs to the resolvent set of matrices $\mathbf{A}$, total values of the growth indices$\vec{x}$can be found as the solution to SLE with matrix $(\mathbf{E}-\delta\mathbf{A}^T)$ and the right-hand side of $\vec{u}$:

$$\vec{x} = (\mathbf{E} - \delta\mathbf{A}^T)^{-1}\vec{u}. \quad (10)$$

If δ^{-1} belongs to the spectrum, then there will be a non-zero solution $\vec{e} \neq 0$ even with zero impact $\vec{u} = \vec{0}$:

$$(\mathbf{E} - \delta\mathbf{A}^T)e = \vec{0}, \quad (11)$$

expressed in terms of the eigenvalues $\lambda_i (1 \leq i \leq n)$ and their corresponding eigenvectors $e_i (1 \leq i \leq n)$ of matrix $\mathbf{A}^T$. In the solution of $\vec{x}$, regardless of the value δ, the summands $\vec{e}_i$ can be allocated. Generally λ_i, and the components $\vec{e}_i$ will be complex numbers even for a real matrix $\mathbf{A}$. Real parts of λ_i correspond to the indices of exponential growth (in the case of a positive number) or decrease (in the case of a negative number), and imaginary parts correspond to their own oscillation frequency. Multiplicity of a root λ_i adds polynomial multipliers.

3.2. Adaptation of the Control Models for Socio-Economic Systems

The fundamental difference between socio-economic systems and technical systems can be described as follows. At first, in the socio-economic systems many events have cyclical, but not a strictly periodic nature (i.e. recurrence occurs), but not in constant time intervals (with the exception of agriculture). This feature is a consequence of the fact that the system is not actually nonlinear, the spectrum is a supplement to the resolvent set, instead of being a finite set of roots of the characteristic polynomial, which will depend on the solutions of x, for the analysis of which it was designed. It means that such analysis would be difficult and there is a risk of misleading.

The second fundamental difference is that the stability of the socio-economic model is not prerequisite condition, as it is for technical devices. For example, the characteristic time of a word pronunciation by a man lasts about 1 second. During this time, the receiver-transmitter of a GSM-module produces about one million oscillations at a frequency of 900/1800/1900 MHz. If the device works unstably (the receiver or the transmitter moves to the generator mode), it will manifest itself in a few tens of oscillations. As a result, the conversation between two users will be impossible. The conversation will also be impossible if the carrier frequency is not stable, that is, the period of oscillation is only approximately unchanged. In the socio-economic systems, a company that seeks to gain a competitive advantage will try to “break the closed cycle” of the stability conditions. Moreover, everyone is aware that aspirations of competitors are the same, and no one gets a competitive advantage forever. Therefore, the problem is established for the next economic cycle, and if you can hold the position for not only one but for several cycles, this is an additional benefit.

Finally, the third difference is that the goals may not be unclear or stated and must be defined as a result of the problem solution. That is, the control impact should be aimed at the indices, on the growth of which the impact can be directed in the certain socio-economic system. It makes no sense to refocus agriculture in Yakutia (Extreme North of Russia) from reindeer herder to a coffee cultivation, if there are favorable world prices for coffee. That is, control impact for agriculture in Yakutia should not be directed by the problem to cultivate coffee, but the most optimal solution should be defined from the socio-economic system itself (the agriculture in Yakutia).

To account all these aforementioned differences it is proposed to change the formulation of the problem, i.e. to determine the vector of the indices growth $\vec{x}$ based not on the known vector of control impacts $\vec{u}$, but by solving the SLE, see Eq. (10), and the vector $\vec{u}$ - from the solution to the optimization problem of the maximization of R: the ratio of the squares of norms $\|\bar{x}\|^2$ and $\|\bar{u}\|^2$:

$$R = \frac{\|\vec{x}\|^2}{\|\vec{u}\|^2} = \frac{((\mathbf{E} - \delta\mathbf{A}^T)^{-1}\vec{u}, (\mathbf{E} - \delta\mathbf{A}^T)^{-1}\vec{u})}{(\vec{u}, \vec{u})},$$

$$= \frac{(\boldsymbol{Cu}, \boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})} \to \max, \tag{12}$$

where $\vec{x} = (x_1 x_2 \dots x_n)^T$ is the vector of indices growth associated with the vertices; $\vec{u} = (u_1 u_2 \dots u_n)^T$ is the vector of control impacts; $\boldsymbol{C} = ((\mathbf{E} - \delta \mathbf{A})^T (\mathbf{E} - \delta \mathbf{A}))^{-1}$ is the symmetric positively definite (or semidefinite) matrix.

After that, the vector $\vec{x}$ is determined by solving SLE, see Eq. (10). In the absence of restrictions on the control impacts $\vec{u}$, the maximum value of $R_{\max}$ is equal to the maximum eigenvalue $\lambda_{\max}$ of the matrix $\mathbf{C}$ and is achieved on the eigenvectors $\vec{e}_{\max}$ of the matrix $\mathbf{C}$, which corresponds to the eigenvalue of $\lambda_{\max}$. From the symmetry of the matrix $\mathbf{C}$ follows, that all its eigenvalues including $\lambda_{\max}$ are real numbers. All components of the eigenvectors are also real numbers. The solution of $\vec{x}$ SLE, see Eq. (10), with matrix and right-hand side containing only real numbers will also consist only of real numbers. This eliminates the problem of economic parameters interpretation expressed in complex numbers because they cannot be expressed this way. Also we add that for positive semidefinite matrix $\mathbf{C}$ all its eigenvalues are non-negative, which is also typical for the values of economic parameters.

Thus, the formulation of the problem, see Eq. (12), consider all these differences. The solution of $\vec{u} = \vec{e}_{\max}$ is not given but is determined by solving the optimization problem. The stability conditions do not dominate the solution $\vec{u} = \vec{e}_{\max}$, and most likely they are even being violated. All are computations are carried out not in the complex, but in the real domain.

The differences in the task formulation are summarized as follows. Solution of $\vec{x}$ SLE, see Eq. (12), is designed for an arbitrary right-hand side of $\vec{u}$ and so it may contain any eigenvectors of matrix $(\mathbf{E} - \delta \mathbf{A}^T)$, which are complex in general case. Solution Eq. (12) is achieved on a concrete real vector $\vec{u} = \vec{e}_{\max}$ of the symmetric matrix $\mathbf{C}$. Therefore, for the real matrix $(\mathbf{E} - \delta \mathbf{A}^T)$, the solution of $\vec{x}$ is real number as well. If the eigenvalues of matrix $(\mathbf{E} - \delta \mathbf{A}^T)$ are outside of the unit circle, the small changes of component $\vec{u}$ will lead to noticeable changes in the vector component $\vec{x}$ (instability). In case of a strong sensitivity to the accuracy of the assignment of $\vec{u}$ the solution cannot be trusted. In the formulation of the problem, see Eq. (12), the concept of the strong changes in output data at small changes in the input data is absent, because the basis of determination $\vec{u} = \vec{e}_{\max}$ and $R_{\max}$ contains converging iterative processes. The stability conditions do not dominate and are even meant to be disordered. In the competitive environment of the socio-economic system there are many reasons that will hinder the realization of the optimal control.

Suppose that the control impact u_j does not change the index x_j, but is transmitted to the indices $x_i (1 \leq i \leq n)$, which are related by arcs a_{ji} with indices x_j:

$$x_j = \sum_{i=1}^{n} a_{ij} (u_i + \delta \sum_{k=1}^{n} a_{ki} x_k), \tag{13}$$

or in matrix form:

$$\vec{x} = \mathbf{A}^T (\vec{u} + \delta \mathbf{A}^T \vec{x}) \text{or} (\mathbf{E} - \delta \mathbf{A}^T \mathbf{A}^T) \vec{x} = \mathbf{A}^T \vec{u}. \tag{14}$$

The model can also be represented as:

$$\begin{aligned} \vec{x} \quad &= \mathbf{A}^T (\vec{u} + \delta (\mathbf{A}^T)^3 \vec{u} + \delta^2 (\mathbf{A}^T)^5 + \\ &+ \dots + \delta^k (\mathbf{A}^T)^{2k+1} \vec{u} + \dots), \end{aligned} \tag{15}$$

in the case of absolute convergence of the resulting series.

The statement of the problem, see Eq. (12), takes the form:

$$\begin{aligned} \boldsymbol{R} \quad &= \frac{\|\vec{x}\|^2}{\|\vec{u}\|^2}, \\ &= \frac{((\mathbf{E} - \delta \mathbf{A}^T \mathbf{A}^T)^{-1} \mathbf{A}^T \vec{u}, (\mathbf{E} - \delta \mathbf{A}^T \mathbf{A}^T)^{-1} \mathbf{A}^T \vec{u})}{(\vec{u}, \vec{u})}, \\ &= \frac{(\mathbf{B} \vec{u}, \vec{u})}{(\vec{u}, u)} \to \max, \end{aligned} \tag{16}$$

where $\mathbf{B} = \mathbf{A}(\mathbf{E} - \delta \mathbf{A}\mathbf{A})^{-1} (\mathbf{E} - \delta \mathbf{A}^T \mathbf{A}^T)^{-1} \mathbf{A}^T = \mathbf{A}(\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{A}^T$ is the symmetric positive definite (semidefinite) matrix.

The problem of determining the optimal control in a linear system with feedback in the general case is reduced to the solution of linear systems. The eigenvalues and eigenvectors of the matrix SLE contain information about the direction of the impact vector, maximizing the amplitude of the vector response to an applied impact, but in general are the complex numbers. To move from the domain of complex numbers to the area of real ones, the formulation of the problem is changed. The ratio of the squares of norms responses and impacts is maximized, leading to the *symmetrization* of the matrix. In this case, not the dynamic oscillatory processes is important to us, but only maximization of the total energy of the control system, including the energy concluded in these oscillatory processes.

Unlike the original matrix in which element a_{ij} meant the arc weight linking vertex v_i with vertex v_j, the element b_{ij} of symmetrized matrix express a *quotient of similarity* of fan-beam structure of the impact transfer of vertices v_i to all other vertices with a similar fan-beam structures of the impact transfer of vertices v_j to all other vertices. Naturally, a measure of this similarity is symmetric. This approach to the symmetrization (for a directed graph) can be named the *gear-ratio symmetrization*. The matrix $\mathbf{B}$ of the quadratic form can be considered as a generalization of the co-citation matrix [15]. After receiving the responses to the optimal control, the question of the sustainability of responses can be raised. Solving this problem the generalization of the bibliographic matrix will appear.

Matrix $\mathbf{B}$ is defined as the product $\mathbf{B} = \mathbf{AF}$, and matrix $\mathbf{F} = (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{A}^T$ is found from the solution of SLE with a unified matrix $(\mathbf{Z}^T\mathbf{Z})$, but with different right-hand sides: j^{th} column of the matrix $\mathbf{F}$ is the solution of linear systems with right-hand side - the j^{th} row of the adjacency matrix $\mathbf{A}$. This SLE is solved by LU-decomposition, as well as other linear systems with non-symmetric matrices. The alternative Cholesky decomposition contains computations of square roots. As a result, a smaller number of operations (taking the root) in Cholesky decomposition in comparison with the number of operations (division) in the LU-decomposition does not mean practically more rapid solution to the problem, while the accumulation of rounding errors during are computing the square root is bigger than during the division operation [16].

If the solution to the problem is not restricted by any conditions, it is equivalent to the problem of finding the maximum eigenvalue of $R_{\max} = \lambda_{\max}$ and corresponding eigenvector $\vec{u} = \vec{e}_{\max}$ of the real symmetric positive (semi-) definite matrix $\mathbf{B}$. Using the function EVCSF of MATH library IMSL all eigenvalues and all eigenvectors of a real symmetric matrix can be found in [17], [18] and [19].

If the components of the control are limited by any conditions, then instead of the solution to the problem of finding the eigenvectors and eigenvalues to determine $\vec{u}_1, \vec{u}_2, \ldots$ and $R_1, R_2, \ldots$, we have to solve the corresponding problem of constrained optimization. Moreover, for further analysis in addition to existing restrictions, the mutual orthogonality of different controls $\vec{u}_i$ and $\vec{u}_j$ (eigenvectors of a real symmetric matrix are orthogonal) may be required. It is expected that we will be interested in a few first most effective controls $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_i$. This decision is made by the user. To support the decision being made regarding the values $R_1, R_2, \ldots R_k$ $(R_1 \geq R_2 \geq \ldots \geq R_k)$, the parameters $(s_2, s_3, \ldots, s_{k-1})$ are computed:

$$s_i = -R_{i-1} + 2R_i - R_{i+1}. \tag{17}$$

Sign change in the sequence s_i indicates the boundary l: $s_{l-1} < 0$, $s_l > 0$, $s_{l+1} < 0$. Other criteria can be used as well. In order to find $\vec{u}_i$ from solution of the SLE, see Eq. (14), we define $\vec{x}_i$.

For each component of each selected control $\vec{u}_i$ we determine ensuring controllability index for the j-th vertex:

$$v_{i,j}^2 = \frac{u_{i,j}^2}{u_{i,1}^2 + u_{i,2}^2 + \ldots + u_{i,n}^2}, \tag{18}$$

and for the components of the corresponding solution $\vec{x}_i$ – effectiveness index for the j-th vertex:

$$r_{i,j}^2 = \frac{x_{i,j}^2}{x_{i,1}^2 + x_{i,2}^2 + \ldots + x_{i,n}^2}, \tag{19}$$

that are then in descending order. The decision on the selection of the most effective components is made by the user. To support the decision the above described or other criteria can be used.

The following Alg. (1) for finding components of effective controls implements the actions mentioned above.

Algorithm 1 Algorithm of effective controls.

Require: $\mathbf{A}$ - adjacency matrix, δ - decrement, ε - stopping treshold $\mathbf{u_0} = (1\,1\,...1)^T$.
Ensure: $\vec{u_1},\ldots,\vec{u_l}$ - vectors of optimal control,
$R_1,\ldots,R_l$ - max objective function R,
$\vec{x_1},\ldots,\vec{x_l}$ - vectors of indices growth associated with the vertices,
$v_{1,1}^2,\ldots,v_{l,n}^2$ - ensuring controllability index,
$r_{1,1}^2,\ldots,r_{l,n}^2$ - effectiveness index.
compute the matrix $\mathbf{Z} \leftarrow \mathbf{E} - \delta\mathbf{AA}$
solve the n systems of linear algebraic equations $\mathbf{F} \leftarrow (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{A}^T$
compute matrix of quadratic form $\mathbf{B} \leftarrow \mathbf{AF}$
$k \leftarrow 0$
repeat
  Gram-Schmidt orthogonalization process [16] $\vec{\hat{u}} \leftarrow \vec{u}_0 - \sum_{i=1}^{k}(\vec{u}_i, \vec{u}_0)\vec{u}_i$
repeat
  update vector $\vec{u} \leftarrow \vec{\hat{u}}$
  compute vector norm $R \leftarrow \|\vec{u}\|$
  compute vector $\vec{\hat{u}} \leftarrow \mathbf{B}\vec{u}/R$
  Gram-Schmidt orthogonalization process $\vec{\hat{u}} \leftarrow \vec{\hat{u}} - \sum_{i=1}^{k}(\vec{u}_i, \vec{\hat{u}})\vec{u}_i$
until $\|\mathbf{\hat{u}} - \mathbf{u}\| < \varepsilon$
  $k \leftarrow k + 1$
  normalize vector $\vec{u_k} \leftarrow \vec{u}/R$
until $k \geq l$
for $i{:} = 1$ **to** n **do**
  solve a system of linear algebraic equations $\vec{x_i} \leftarrow (\mathbf{Z}^T)^{-1}\mathbf{A}^T\vec{u_i}$
  compute vector norm $\|\vec{x}_i\| \leftarrow \sqrt{\sum_{j=1}^{n} x_{i,j}^2}$
for $i := 1$ **to** n **do**
  compute $v_{i,j}^2 \leftarrow u_{i,j}^2/R_i; r_{i,j}^2 \leftarrow x_{i,j}^2/\|\vec{x}_i\|$
end for
  Sort $v_{i,j}^2$ so that $v_{i,j(1)}^2 \geq v_{i,j(2)}^2 \cdots v_{i,j(n)}^2$
  Sort $r_{i,j}^2$ so that $r_{i,j[1]}^2 \geq r_{i,j[2]}^2 \cdots r_{i,j[n]}^2$
end for
return $\vec{u_1},\ldots,\vec{u_l}$; $R_1,\ldots,R_l$; $\vec{x_1},\ldots,\vec{x_l}$; $v_{1,1}^2,\ldots,v_{l,n}^2$; $r_{1,1}^2,\ldots,r_{l,n}^2$.

3.3. Metrics of the Vertices on the Basis of the Theory of Effective Controls

On the basis of the above-described algorithm, we consider the proposed metrics of vertices characterizing the measure of effective controls.

Definition 1. $Ir_{i,j}$ *- Growth index - parameter of the growth of model factor corresponding to the j-th vertex of the graph under the directive impact* Ia_i *in general.*

Growth index $Ir_{i,j}$ j^{th} vertex under the i^{th} impact is defined as the solution of SLE, see Eq. (14):

$$(\mathbf{E}-\delta\mathbf{A}^T\mathbf{A}^T)Ir_{i,j}=\mathbf{A}^T Ia_{i,j}, \qquad (20)$$

where δ is the damping factor, $\mathbf{A}$ is the adjacency matrix of the graph G, and $Ia_{i,j}$ is the control index.

Definition 2. $Ia_{i,j}$*- Control index - coefficient, with which the index of* j^{th} *vertex of the graph participates in the directive control.*

Control index $Ia_{i,j}$ is found as a result of solving the optimization problem Eq. (16):

$$\begin{aligned}\boldsymbol{R} &= \frac{\|Ir_{i,j}\|^2}{\|Ia_{i,j}\|^2} \\ &= \frac{((\mathbf{E}-\delta\mathbf{A}^T\mathbf{A}^T)^{-1}\mathbf{A}^T Ia_{i,j},(\mathbf{E}-\delta\mathbf{A}^T\mathbf{A}^T)^{-1}\mathbf{A}^T Ia_{i,j})}{(Ia_{i,j},Ia_{i,j})} \\ &= \frac{(BIa_{i,j},Ia_{i,j})}{(Ia_{i,j},Ia_{i,j})}\to \max, \qquad (21)\end{aligned}$$

where $\mathbf{B}=\mathbf{A}(\mathbf{E}-\delta\mathbf{A}\mathbf{A})^{-1}(\mathbf{E}-\delta\mathbf{A}^T\mathbf{A}^T)^{-1}\mathbf{A}^T=\mathbf{A}(\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{A}^T$ is the symmetric positive (semi-) definite matrix; $\mathbf{Z}=\mathbf{E}-\delta \boldsymbol{A}\boldsymbol{A}$.

Definition 3. $r^2_{i,j}$*- Effectiveness index the fraction of conformity of growth vector* $\vec{x}_i$ *alignment selected by the component corresponding to the* j^{th} *vertex.*

Effectiveness index is are computed by the Eq. (19) as the ratio of the square of the j-th component of the growth vector $\vec{x}_i$ to the sum of the squares of all the components of the growth vector $\vec{x}_i$:

$$r^2_{i,j}=\frac{x^2_{i,j}}{x^2_{i,1}+x^2_{i,2}+...+x^2_{i,n}}. \qquad (22)$$

Definition 4. $v^2_{i,j}$*- Ensuring controllability index - the fraction of conformity of control vector* $\vec{u}_i$*due to the index of j-th vertex.*

Ensuring controllability index is are computed by the Eq. (18) as the ratio of the square of the j-th component of the control vector $\vec{x}_i$ to the sum of the squares of all the components of the control vector $\vec{u}_i$:

$$v^2_{i,j}=\frac{u^2_{i,j}}{u^2_{i,1}+u^2_{i,2}+...+u^2_{i,n}}. \qquad (23)$$

4. Experiment

Consider the procedure of finding the productive components of effective controls on the example of the cognitive model as a directed weighted graph with cause-and-effect relationships, represented by the adjacency matrix of dimension 75×75 with 411 arcs and their total weight 216.5 [20].

An attempt of clustering by minimizing the quadratic functional led to next result – all vertices form a single cluster. Trying to identify the ordering relationship of order in the graph and to raise the question of exclusion of the arcs, which form cycles, led to the necessity to remove about a quarter of the arcs, which distort the essence of the graph. The intention to compute the mutual influences of the vertices to each other in terms of the existence of multiple cycles leads to the need for an introduction of damping factor, while the influence will be determined from the SLE. SLE appears to be ill-conditioned, but solvable even if the value of the damping factor $\delta=1$. It was also found that the resonance properties of the SLE matrix, see Eq. (10), determine that the eigenvalues and eigenvectors are complex. However, due to the fact that the elements of the matrix and the right part of the SLE are real numbers, eigenvalues and eigenvectors of the matrix form a complex conjugate pairs. Nevertheless, interpretation of economic indicators expressed by complex numbers is challenging. Therefore, we consider the optimization problem, which has a real solution. Solution Eq. (16) is not problematic except that the condition number of matrix 75×75 reach $\sim 10^{15}$. The results of computation of the components of the eigenvector $\mathbf{u_1}$ of matrix $\mathbf{B}$ (paragraph 5-15 of Alg. (1), are presented in Tab. 1 (sorted in descending order).

Solution of SLE, see Eq. (14), (paragraph 16-18 of Alg. 1) is presented in the Tab. 2 (sorted in descending order).

After determining the first main direction $\vec{u}_i$, the contribution of each individual parameter was defined corresponding to the vertex of the graph, to every single vector component of the computed parameters $\vec{x}_i$, i.e. it was decided to compute the elements of the inverse matrix. The resulting matrix with a very high accuracy coincides with the dyadic product of two vectors for SLE, see Eq. (14):

$$(\mathbf{E}-\delta\mathbf{A}^T\mathbf{A}^T)^{-1}\mathbf{A}\approx\tilde{\mathbf{x}}\cdot\tilde{\mathbf{u}}^T. \qquad (24)$$

For the optimization problem Eq. (16):

$$\mathbf{B}\approx(\vec{u}\cdot\vec{x}^T)\cdot(\vec{x}\cdot\vec{u}^T)=\vec{u}\cdot R_{\max}\cdot\vec{u}^T. \qquad (25)$$

That is, the matrix rows differ from each other only by a multiplier associated with the row of the matrix and the columns of the matrix are also different from

Tab. 1: The values of the vector components $\vec{u}_1$, corresponding to the vertices v_i.

v_n	u_{1j}	v_n	u_{1j}	v_n	u_{1j}
5	0.2528038	29	0.05987265	16	-0.03436201
12	0.2526091	14	0.05542509	35	-0.03447989
39	0.2101528	43	0.05301107	36	-0.03983346
34	0.2099352	13	0.04879709	64	-0.04155279
11	0.2008675	32	0.04752683	68	-0.04254057
2	0.1876696	44	0.04523298	52	-0.04260134
75	0.1657376	27	0.03099309	10	-0.04923852
38	0.158184	59	0.02701188	40	-0.05052563
1	0.1547691	56	0.02440797	65	-0.06068351
54	0.1475701	63	0.01614966	72	-0.07647701
25	0.1447973	60	0.01389593	4	-0.08341967
6	0.1424852	26	0.01282121	23	-0.0835738
50	0.1195439	57	0.01074903	15	-0.09434044
55	0.111834	17	0.009989827	31	-0.1069596
20	0.09923731	8	0.007004421	7	-0.1195215
47	0.09516039	21	0	30	-0.1223333
37	0.0935998	28	0	46	-0.1252972
45	0.09050892	51	0	48	-0.1402221
69	0.08868004	70	-0.002065489	42	-0.1831398
9	0.08566853	71	-0.002968655	22	-0.1919131
41	0.08098736	74	-0.004202652	19	-0.1935265
67	0.07589548	61	-0.007046627	3	-0.1951359
18	0.07455603	33	-0.02530296	66	-0.220606
58	0.0739199	73	-0.03073383	49	-0.2269727
62	0.06634766	53	-0.03408115	24	-0.2691427

Tab. 2: The values of the vector components $\vec{x}_1$, corresponding to the vertices v_i, for control $\vec{u}_1$.

v_n	u_{1j}	v_n	u_{1j}	v_n	u_{1j}
37	208.0518	34	5.794519	28	-10.2296
47	183.2326	14	5.574492	70	-10.8209
51	163.3114	12	4.614484	22	-11.8735
41	148.3645	8	4.193422	32	-12.0834
26	102.8048	42	2.408943	48	-12.721
25	94.21774	24	1.916675	3	-15.9206
30	91.09594	56	1.110261	49	-17.279
75	86.72815	20	0.548213	7	-18.1356
52	78.78956	66	0.083993	23	-23.5826
50	67.93214	72	0.028935	38	-27.4331
27	66.40107	61	0.02748	44	-38.3505
39	48.42215	65	0.022907	35	-38.5159
36	44.63304	64	0.01561	6	-39.4433
1	32.46423	73	0.011676	21	-39.5259
31	27.01683	74	0.001598	18	-39.5922
40	21.58144	71	0.000984	15	-40.2072
57	14.47225	63	-0.00615	17	-47.0725
11	12.33464	13	-0.01818	54	-52.3909
68	11.62518	29	-0.02245	53	-53.2926
5	10.83454	67	-0.0286	46	-67.0565
10	10.48386	16	-1.98022	60	-73.45
62	10.04127	9	-3.08709	4	-73.7708
59	9.481671	33	-3.47675	58	-80.345
2	7.127377	69	-6.55189	43	-87.569
19	6.409531	55	-7.77439	45	-120.328

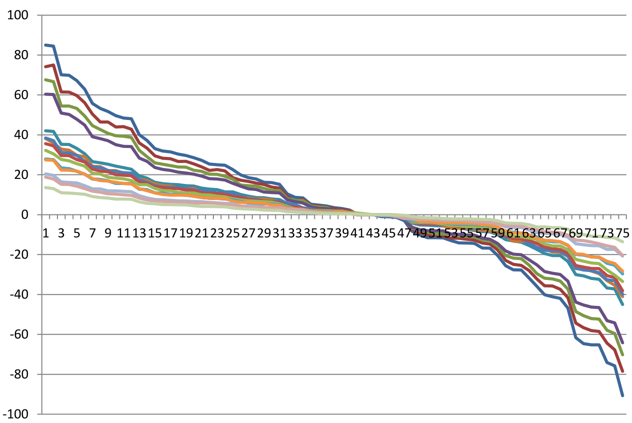


Fig. 1: Collinearity of vectors, expressed by ordered components of the rows of the matrix of dyadic product.

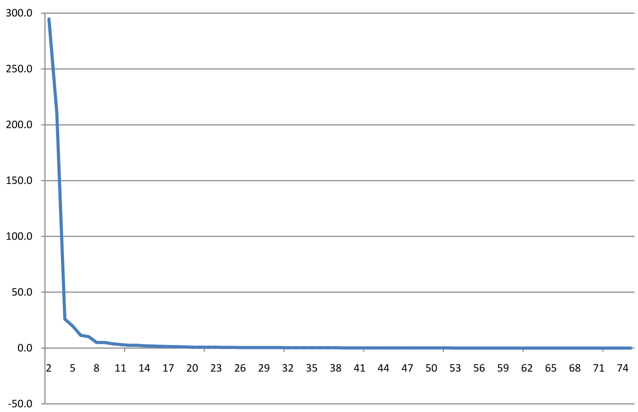


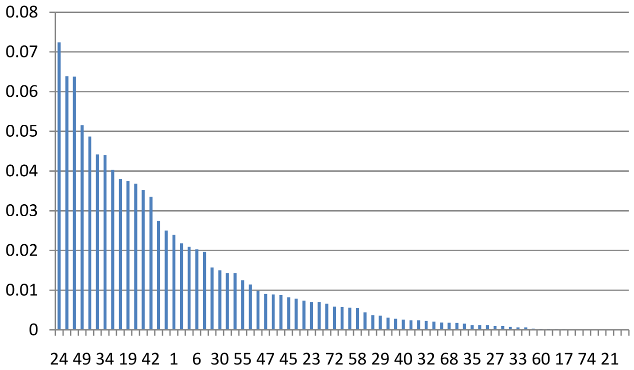
Fig. 2: Sorting of the values $R_1, R_2, \ldots, R_n$ (without considering the maximum eigenvalue).

each other only by a multiplier associated with a column. If the elements of rows or elements of columns are drawn graphically, they will represent a family of broken lines similar to each other up to a scale multiplier. A better visual representation is obtained if we pre-arrange the elements of rows or columns (note that due to the strong collinearity they acquire the same order) (Fig. 1).

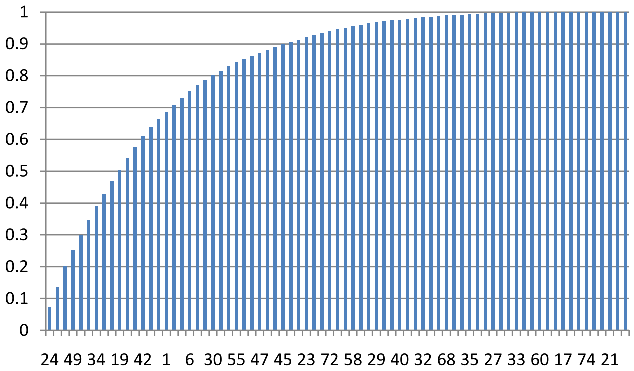
Figure 1 shows the corresponding picture of the ordering in descending manner for the row of dyadic product (a similar picture for the columns), moreover it coincides with the ordered descending components of solution (Fig. 2). After that, it was decided to determine the subsequent eigenvalues.

It turns out that if we interpret the matrix of the quadratic form as the covariance matrix, the operation of the system is determined by one leading development direction, which greatly prevails over the other. This explains why communities were not allocated in the graph on the basis of minimizing the quadratic functional.

Sorting of the values $R_1, R_2, \ldots, R_n$ shows a significant gap of the first vector from the second $(3 \cdot 10^3)$. From the structure of the matrix Eq. (24), it follows that strong interrelation of parameters action on any of them leads to the same vector of growth, however with different scales. Then we only need to select the one controllable parameter (maximum). For more effective control we decided to select three parameters, which are apart from the others (Fig. 3(a), Tab. 3). Figure 3(b) shows the effect of the accumulation of ensuring controllability index $v_{i,j}^2$.



(a) Ensuring controllability index ordering.



(b) Integral representation of Ensuring controllability index.

Fig. 3: Ensuring controllability index $v_{i,j}^2$.

Tab. 3: Selected vertices and their corresponding components of Ensuring controllability index.

x_l	$v_{i,j}^2$	Integral $v_{i,j}^2$
x_{24}	0.072438	0.072438
x_5	0.06391	0.136348
x_{12}	0.063811	0.200159

Ordered components of the growth vector of index $\vec{x}_i$ are presented in Fig. 4. Impact on any parameter (vertex) leads to this growth vector but with its own scale. Descending components of this vector can be interpreted as the increase of the determination coefficient. Thus, the increase of the determination coefficient to values above 0.5 is provided by the first few components.

Figure 5(a) shows the corresponding Effectiveness indexes, and Fig. 5(b) - their integral representation.

The first four components provide more than 50 % of the maximum growth (Tab. 4).

Summarizing, the presented result expresses the choice of effective controls for the socio-economic system, provided by the cognitive model.

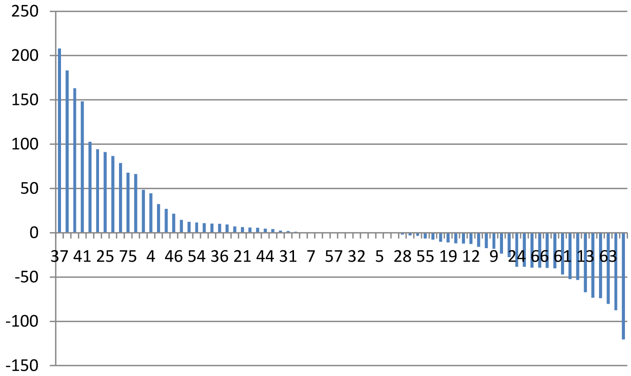


Fig. 4: Ordered components of the growth vector of index $\mathbf{x_i}$.

Tab. 4: Selected vertices and corresponding components of Effectiveness index.

x_l	$r_{i,j}^2$	Integral $r_{i,j}^2$
x_{37}	0,174729	0,174729
x_{47}	0,135527	0,310256
x_{51}	0,10766	0,417916
x_{41}	0,088855	0,506771

5. Discussion of the Results

Five different criteria for the evaluation of our approach may be applied:

1) Applicability in the Subject Area

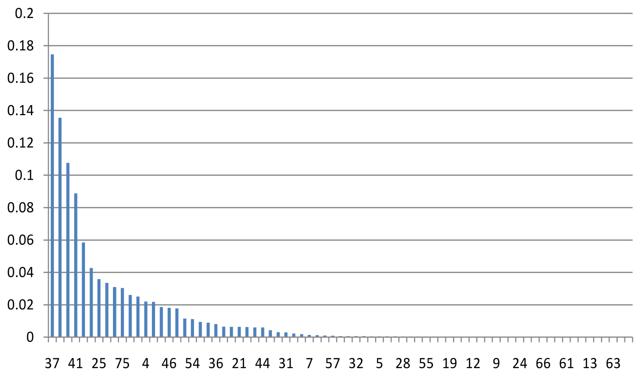
The conducted study had shown the applicability of the developed approach based on the systems theory for the analysis of directed signed weighted graphs with cause-and-effect relationships, which represent cognitive models in social and economic systems

2) Feasibility

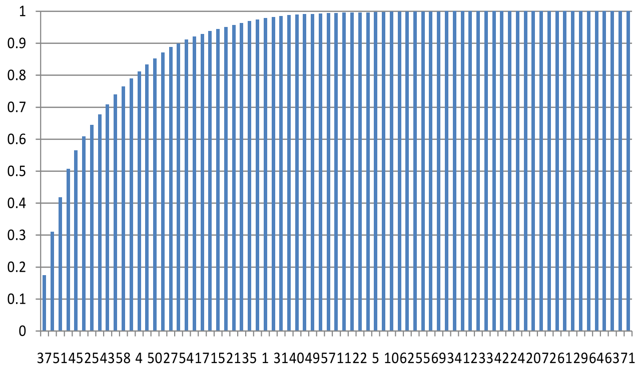
SLE solutions are uniquely solvable, because the matrix $(\mathbf{Z}^\mathbf{T}\mathbf{Z})$ is positive semidefinite by its construction and can be made a strictly positive definite, and thus a non-degenerate by choosing the decrement δ. The feasibility of the solution to an optimization problem for a relationship of the squares of norms follows from the convergence of the corresponding iterative processes. The non-degeneracy of the matrix SLE, see Eq. (14), is provided by a choice of the damping factor δ.

3) Format of the Results

As a result of the applied algorithm the selection of effective components and the corresponding effective significant impacts, which represent key vertices that display overall system, is carried out. The quantity of



(a) Effectiveness index ordering.



(b) Integral representation of Effectiveness index.

Fig. 5: Effectiveness index $r^2_{i,j}$.

the obtained vertices is reduced to a reasonable number of vertices and arcs that allows to reach an acceptable level of information obtained by the graph model for an expert (decision maker).

4) Difficulty and Complexity of the Solution

Solution time of algorithm takes a fraction of a second. The computational complexity of the algorithm is $O(n^3)$ of arithmetic operations.

5) Quality

The quality of the results provided by the possibility of conversion of solving problems to the systems with symmetric matrices, to which the convergence of the corresponding iterative processes is proven.

6. Conclusion

The proposed method of maximizing the spread of influence in models of socio-economic systems, which can be presented with directed signed weighted cyclic graphs with cause-and-effect relationships, is a reliable solution for the development of practical applications of expert systems and knowledge based systems. Solutions based on the theory of effective controls can be applied to many optimization problems in the development of intelligent systems. Our method can reduce management costs under the conditions of limited opportunities through optimal choice of accompanying impacts without loss of the key factors.

Maximization of a sign-definiteness quadratic form is the key step in the search for the optimal control that ensures the validity of the solution (the existence, possibility and uniqueness of the solutions), computation in a real, rather than in the complex domain, and independence from the conditions of stability.

Development of Algorithm of effective controls is not trivial problem in the new context, as we have changed the formulation of the optimal control problem itself. The vector of control impacts $\vec{u}$ is determined from solving the optimization problem of maximizing the objective function. This approach differs from the traditional solutions, in which the growth vector of vertices characteristics $\vec{x}$ is defined by the known vector of control impacts $\vec{u}$ from solving the SLE, see Eq. (10). The method is applicable for a directed signed weighted graph with cause-and-effect relationships. Notably, this approach does not impose requirements for graphs to order relationship or to have a matrix of a special kind. Moreover, fulfilment of the stability conditions are not required.

Many important directions remain for future work. Statement of the problem can be generalized to constrained optimization, i.e. solutions satisfying certain restrictions. However, as the directions that differ only by the factor represent the same direction, the restrictions should not express the conditions for the components of the desired vectors, but belong to some unknown vector (linear) manifold which is generated by a given set of vectors. A promising area could be the expansion of the concept of equivalence of the graph (the conditional equivalence). Application of the theory of effective controls, based on the provisions of the general systems theory and systems analysis in the annex to the management in socio-economic systems, allows to identify the components of productive fuzzy cognitive model of the system and transform them into a graph of conditional equivalence.

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References

[1] KOSKO, B. Fuzzy cognitive maps. *International Journal Man-Machine Studies*. 1986, vol. 24, iss. 1, pp. 65–75. ISSN 0020-7373.

[2] SCOTT, J. *Social network analysis*. London: Sage Publications, 2000. ISBN 0-7619-6338-3.

[3] ALBERT, R. and A. L. BARABASI. Statistical mechanics of complex networks. *Reviews of Modern Physics*. 2002, vol. 74, iss. 1, pp. 47–97. ISSN 0034-6861. DOI: 10.1103/RevModPhys.74.47.

[4] NEWMAN, M. E. J. The structure and function of complex networks. *Society for Industrial and Applied Mathematics - SIAM Review*. 2003, vol. 45, iss. 2, pp. 167–256. ISSN 0036-1445. DOI: 10.1137/S003614450342480.

[5] NEWMAN, M. E. J. *Networks: An Introduction*. New York: Oxford University Press, 2010. ISBN 978-0199206650.

[6] ZHAO, S. X. and F. Y. YE. Exploring the directed h-degree in directed weighted networks. *Journal of Infometrics*. 2012, vol. 6, iss. 4, pp. 619–630. ISSN 1751-1577. DOI: 10.1016/j.joi.2012.06.007.

[7] ZHANG, J. Y., Z. Q. LIU and S. ZHOU. Quotient FCMs -a decomposition theory for tuzzy cognitive maps. *IEEE Transactions on fuzzy systems*. 2003, vol. 11, iss. 5, pp. 593–604. ISSN 1941-0034. DOI: 10.1109/TFUZZ.2003.817836.

[8] ROMANENKO, V. D. and Y. L. MILYAVSKIY. Ensuring the sustainability of pulse processes in cognitive maps on the basis of the models in the states space. *System Research & Information Technologies*. 2014, vol. 1, iss. 1, pp. 26–42. ISSN 1681-6048.

[9] LIU, L., X. CHEN, M. LIU, Y. JIA, J. ZHONG, R. GAO and Y. ZHAO. An influence power-based clustering approach with PageRank-like model. *Applied Soft Computing*. 2016, vol. 40, iss. 1, pp. 17–32. ISSN 1568-4946. DOI: 10.1016/j.asoc.2015.10.050.

[10] KEMPE, D., J. KLEINBERG and E. TARDOS. Maximizing the spread of influence through a social network. In: *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*. Washington, D.C.: ACM New York, 2003, pp. 137–146. ISBN 1-58113-737-0. DOI: 10.1145/956750.956769.

[11] JIANG, Q., G. SONG, G. CONG, Y. WANG, W. SI and K. XIE. Simulated Annealing Based Influence Maximization in Social Networks. In: *25th AAAI Conference on Artificial Intelligence and the 23rd Innovative Applications of Artificial Intelligence Conference*. San Francisco: American Association for Artificial Intelligence (AAAI) Press, 2011, pp. 127–132. ISBN 978-157735508-3.

[12] RICHARDSON, M. and P. DOMINGOS. Mining Knowledge-Sharing Sites for Viral Marketing. In: *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. Edmonton: ACM, 2002, pp. 61–70. ISBN 1-58113-567-X. DOI: 10.1145/775047.775057.

[13] ZENG, Y., X. CHEN, G. CONG, S. QIN, J. TANG and Y. XIANG. Maximizing influence under influence loss constraint in social networks. *Expert Systems with Applications*. 2016, vol. 55, iss. 1, pp. 255–267. ISSN 0957-4174. DOI: 10.1016/j.eswa.2016.01.008.

[14] MARINAZZO, D., M. PELLICORO and S. STRAMAGLIA. Kernel–Granger causality and the analysis of dynamical networks. *Physical Review E*. 2008, vol. 77, iss. 5, pp. 056215–056224. ISSN 1539-3755. DOI: 10.1103/PhysRevE.77.056215.

[15] MALLIAROS, F. D. and M. VAZIRGIANNIS. Clustering and community detection in directed networks: A survey. *Physics Reports*. 2013, vol. 533, iss. 4, pp. 95–142. ISSN 0370-1573. DOI: 10.1016/j.physrep.2013.08.002.

[16] HORN, R. A. and C. R. JOHNSON. *Matrix analysis*. New York: Cambridge University Press, 2013. ISBN 978-0-521-83940-2.

[17] SMITH, B. T., J. M. BOYLE, J. J. DONGARRA, B. S. GARBOW, Y. IKEBE, V. C. KLEMA and C. B. MOLER. *Matrix Eigensystem Routines - EISPACK Guide*. New York: Springer-Verlag, 1976. ISBN 978-3-540-07546-2.

[18] PARLETT, B. N. *The Symmetric Eigenvalue Problem*. Englewood Cliffs: Prentice-Hall, 1980. ISBN 0-13-880047-2.

[19] Fortran Numerical Library. *RogueWave* [online]. 1990. Available at: `http://www.roguewave.com/products-services/imsl-numerical-libraries/fortran-libraries`.

[20] TSELYKH, A., V. VASILEV and L. TSELYKH. Fuzzy Graphs Clustering with Quality Relation Functionals in Cognitive Models. In: *Proceedings of the 1st International Scientific Conference on Intelligent Information Technologies for Industry*. Sochi: Springer, 2016, pp. 349–360. ISBN 978-3-319-33609-1. DOI: 10.1007/978-3-319-33609-1_32.

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